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TABLES OF THE ASYMPTOTIC DISTRIBUTION OF THE THIRD EXTREME *

by

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1. Introduction

The density function of the asymptotic distribution of the m-th largest value x_m for initial distributions of the exponential type is given by ([2], page 187):

$$(1) \qquad \varphi_m(x_m) = \frac{m^m}{\Gamma(m)} \exp\left[-m\,\alpha_m(x_m - U_m) - m e^{-\alpha_m(x_m - U_m)} \right]$$

The quantities U_m and α_m are obtained from the probability function $F(x)$, the density function $f(x)$ of the initial distribution and the sample size n as follows:

$$(2) \qquad \begin{aligned} F(U_m) &= 1 - m/n \\ f(U_m) &= m\,\alpha_m/n \end{aligned}$$

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† Deceased.

If we introduce the transformation $y_m = \alpha_m(x_m - U_m)$ the density function of y_m becomes:

$$(3) \qquad \varphi_m(y_m) = \frac{m^m}{\Gamma(m)} \exp\left[-m y_m - m e^{-y_m}\right]$$

which does not contain any parameter. For $m = 1$, equation (3) reduces to the density of the double exponential distribution. For $m = 2$, and 3, which we will call the second and the third extremes, respectively, the equation reduces to

$$(4) \qquad \varphi_2(y_2) = 4 \exp\left[-2y_2 - 2e^{-y_2}\right]$$

$$(5) \qquad \varphi_3(y_3) = (27/2)\exp\left[-3y_3 - 3e^{-y_3}\right] \quad .$$

The corresponding probability functions can be obtained from ([2], page 189) as

$$(6) \qquad \Phi_2(y) = e^{-2e^{-y}}(1 + 2e^{-y})$$

$$\Phi_3(y) = e^{-3e^{-y}}(1 + 3e^{-y} + (9/2)e^{-2y}) \, .$$

The National Bureau of Standards [4] has published tables of the probability functions $\Phi_m(y)$ for certain selected values of y. Gumbel and Greenwood [3] have prepared extensive tables for $\Phi_2(y)$. The object of the present paper is to compute similar tables for $\Phi_3(y)$ (Table 1). We shall also construct probability paper on the basis of this table and use it to analyze a numerical example. This will illustrate the practical motiva-

tion of the present study.

In addition, tables of the densities $\varphi_2(y)$ and $\varphi_3(y)$, given in equations (4) and (5), have also been computed (Table 2).

2. Numerical Example

It is a well known experimental fact that the intervals between the successive radioactive emissions are independently exponentially distributed random variables. Therefore, the largest, the second largest, and the third largest values among n intervals will follow the asymptotic distributions of the largest, the second largest and the third largest values, respectively, provided n is sufficiently large. We consider 20 such first, second and third extremes each based on 400 intervals [1]. These are plotted on the corresponding probability papers (Figures 1, 2, 3). These figures give ample evidence that our assumptions are correct.

In order to fit the theoretical lines, we estimate the parameters α_m and U_m by the method of moments. The theoretical means and variances $\overline{y}_m$ and σ^2_m of the reduced variables y_m can be obtained by calculating its generating function. These are given for $m = 1$ to $m = 10$ in [2], page 194. If $\bar{x}_m$ and s_m are the mean and the standard deviation of the observed m-th extreme x_m ($m = 1, 2, 3$), the estimates of α_m and U_m are given by

$$(7) \qquad 1/\hat{\alpha}_m = s/\sigma_m$$

(8) $\hat{U}_m = \bar{x}_m - \bar{y}_m / \hat{a}_m$

The results of this estimation are summarized in Table 3. The fitted lines are drawn in the corresponding figures. The fit in each case is quite satisfactory.

References

[1] Gumbel, E. J. Les intervalles extrêmes entre les emission radioactives II. Le Journal de Physique et le Radium (1937), pp. 446 - 452.

[2] Gumbel, E. J. Statistics of Extremes, Columbia University Press (1958).

[3] Gumbel, E. J. Greenwood, J.A. Table of the asymptotic distribution of the second extreme. Annals of Mathematical Statistics (1951), pp. 121 - 124.

[4] National Bureau of Standards Probability Tables for the Analysis of Extreme-Value Data. Applied Mathematics Series No. 22 (1953).

Table 1 Probability Function $\Phi_3(y_3)$ of the Third Extreme y_3 .

y_3	$\Phi_3(y_3)$	y_3	$\Phi_3(y_3)$	y_3	$\Phi_3(y_3)$
-1.75	.00001	0.10	.49008	1.95	.99056
-1.70	.00001	0.15	.52293	2.00	.99175
-1.65	.00002	0.20	.55510	2.05	.99279
-1.60	.00004	0.25	.58641	2.10	.99371
-1.55	.00008	0.30	.61670	2.15	.99451
-1.50	.00015	0.35	.64584	2.20	.99522
-1.45	.00027	0.40	.67371	2.25	.99584
-1.40	.00045	0.45	.70024	2.30	.99637
-1.35	.00075	0.50	.72537	2.35	.99685
-1.30	.00120	0.55	.74906	2.40	.99726
-1.25	.00188	0.60	.77129	2.45	.99762
-1.20	.00286	0.65	.79207	2.50	.99793
-1.15	.00425	0.70	.81141	2.55	.99820
-1.10	.00617	0.75	.82935	2.60	.99844
-1.05	.00876	0.80	.84592	2.65	.99865
-1.00	.01218	0.85	.86118	2.70	.99882
-0.95	.01661	0.90	.87519	2.75	.99898
-0.90	.02223	0.95	.88800	2.80	.99912
-0.85	.02922	1.00	.89968	2.85	.99923
-0.80	.03776	1.05	.91031	2.90	.99934
-0.75	.04802	1.10	.91995	2.95	.99943
-0.70	.06015	1.15	.92868	3.00	.99950
-0.65	.07428	1.20	.93655	3.05	.99957
-0.60	.09048	1.25	.94364	3.10	.99963
-0.55	.10880	1.30	.95001	3.15	.99968
-0.50	.12926	1.35	.95573	3.20	.99972
-0.45	.15181	1.40	.96084	3.25	.99976
-0.40	.17636	1.45	.96541	3.30	.99979
-0.35	.20279	1.50	.96948	3.35	.99982
-0.30	.23093	1.55	.97310	3.40	.99984
-0.25	.26059	1.60	.97632	3.45	.99987
-0.20	.29154	1.65	.97918	3.50	.99988
-0.15	.32354	1.70	.98171	3.55	.99990
-0.10	.35632	1.75	.98394	3.60	.99991
-0.05	.38963	1.80	.98592	3.65	.99993
+0.00	.42319	1.85	.98767	3.70	.99994
0.05	.45676	1.90	.98920	3.75	.99994

Table 1 (continued)

y_3	$\Phi_3(y_3)$	y_3	$\Phi_3(y_3)$	y_3	$\Phi_3(y_3)$
3.80	.99995	4.10	.99998	4.40	.99999
3.85	.99996	4.15	.99998	4.45	.99999
3.90	.99996	4.20	.99999	4.50	.99999
3.95	.99997	4.25	.99999	4.55	.99999
4.00	.99997	4.30	.99999	4.60	1.00000
4.05	.99998	4.35	.99999		

Table 2 Density functions $\varphi_2(y)$ and $\varphi_3(y)$ of the Second and Third Extremes

y	$\varphi_2(y)$	$\varphi_3(y)$	y	$\varphi_2(y)$	$\varphi_3(y)$
-2.15	.00001		-0.75	.25984	.22351
-2.10	.00002		-0.70	.28903	.26222
-2.05	.00004		-0.65	.31829	.30303
-2.00	.00008		-0.60	.34717	.34519
-1.95	.00016		-0.55	.37523	.38788
-1.90	.00028	.00001	-0.50	.40206	.43021
-1.85	.00048	.00002	-0.45	.42727	.47131
-1.80	.00081	.00004	-0.40	.45052	.51029
-1.75	.00133	.00008	-0.35	.47150	.54634
-1.70	.00211	.00016	-0.30	.48996	.57875
-1.65	.00325	.00031	-0.25	.50573	.60690
-1.60	.00489	.00058	-0.20	.51866	.63033
-1.55	.00718	.00103	-0.15	.52868	.64868
-1.50	.01028	.00176	-0.10	.53577	.66178
-1.45	.01441	.00292	-0.05	.53997	.66957
-1.40	.01976	.00469	0.00	.54134	.67213
-1.35	.02655	.00730	0.05	.54001	.66965
-1.30	.03500	.01105	0.10	.53613	.66244
-1.25	.04530	.01627	0.15	.52987	.65088
-1.20	.05762	.02334	0.20	.52144	.63540
-1.15	.07207	.03265	0.25	.51104	.61649
-1.10	.08874	.04461	0.30	.49890	.59466
-1.05	.10764	.05959	0.35	.48525	.57042
-1.00	.12870	.07791	0.40	.47032	.54429
-0.95	.15181	.09982	0.45	.45432	.51675
-0.90	.17677	.12542	0.50	.43746	.48826
-0.85	.20332	.15470	0.55	.41996	.45925
-0.80	.23113	.18751	0.60	.40199	.43010

Table 2 (continued)

y	$\varphi_2(y)$	$\varphi_3(y)$	y	$\varphi_2(y)$	$\varphi_3(y)$
0.65	.38374	.40114	2.80	.01310	.00253
0.70	.36536	.37267	2.85	.01192	.00220
0.75	.34700	.34493	2.90	.01085	.00191
0.80	.32878	.31813	2.95	.00987	.00165
0.85	.31082	.29242	3.00	.00898	.00143
0.90	.29322	.26793	3.05	.00816	.00124
0.95	.27605	.24475	3.10	.00742	.00108
1.00	.25938	.22292	3.15	.00674	.00093
1.05	.24327	.20248	3.20	.00613	.00081
1.10	.22776	.18343	3.25	.00557	.00070
1.15	.21289	.16576	3.30	.00505	.00061
1.20	.19867	.14943	3.35	.00459	.00052
1.25	.18513	.13441	3.40	.00417	.00045
1.30	.17226	.12064	3.45	.00378	.00039
1.35	.16006	.10806	3.50	.00343	.00034
1.40	.14854	.09661	3.55	.00312	.00029
1.45	.13768	.08621	3.60	.00283	.00025
1.50	.12746	.07679	3.65	.00257	.00022
1.55	.11787	.06829	3.70	.00233	.00019
1.60	.10888	.06063	3.75	.00211	.00016
1.65	.10048	.05375	3.80	.00191	.00014
1.70	.09264	.04758	3.85	.00174	.00012
1.75	.08533	.04206	3.90	.00157	.00011
1.80	.07853	.03713	3.95	.00143	.00009
1.85	.07221	.03274	4.00	.00129	.00008
1.90	.06635	.02884	4.05	.00117	.00007
1.95	.06092	.02537	4.10	.00106	.00006
2.00	.05589	.02230	4.15	.00096	.00005
2.05	.05124	.01957	4.20	.00087	.00004
2.10	.04695	.01717	4.25	.00079	.00004
2.15	.04299	.01504	4.30	.00072	.00003
2.20	.03935	.01317	4.35	.00065	.00003
2.25	.03599	.01152	4.40	.00059	.00002
2.30	.03290	.01007	4.45	.00053	.00002
2.35	.03006	.00880	4.50	.00048	.00002
2.40	.02746	.00768	4.55	.00044	.00002
2.45	.02506	.00670	4.60	.00040	.00001
2.50	.02287	.00584	4.65	.00036	.00001
2.55	.02086	.00508	4.70	.00032	.00001
2.60	.01902	.00443	4.75	.00029	.00001
2.65	.01734	.00385	4.80	.00027	.00001
2.70	.01579	.00335	4.85	.00024	.00001
2.75	.01439	.00291	4.90	.00022	

Table 2 (continued)

y	$\varphi_2(y)$	$\varphi_3(y)$	y	$\varphi_2(y)$	$\varphi_3(y)$
4.95	.00020		5.90	.00003	
5.00	.00018		5.95	.00003	
5.05	.00016		6.00	.00002	
5.10	.00015		6.05	.00002	
5.15	.00013		6.10	.00002	
5.20	.00012		6.15	.00002	
5.25	.00011		6.20	.00002	
5.30	.00010		6.25	.00001	
5.35	.00009		6.30	.00001	
5.40	.00008		6.35	.00001	
5.45	.00007		6.40	.00001	
5.50	.00007		6.45	.00001	
5.55	.00006		6.50	.00001	
5.60	.00005		6.55	.00001	
5.65	.00005		6.60	.00001	
5.70	.00004		6.65	.00001	
5.75	.00004		6.70	.00001	
5.80	.00004		6.75	.00001	
5.85	.00003				

Table 3 Estimation of the Parameters U_m, $1/\alpha_m$ by Method of Moments

m	Observed Mean	Observed Standard Deviation	U_m	$1/\alpha_m$
1	33.730	6.486	30.8112	5.0569
2	25.825	2.885	24.8523	3.5971
3	23.270	3.048	22.4170	4.8520

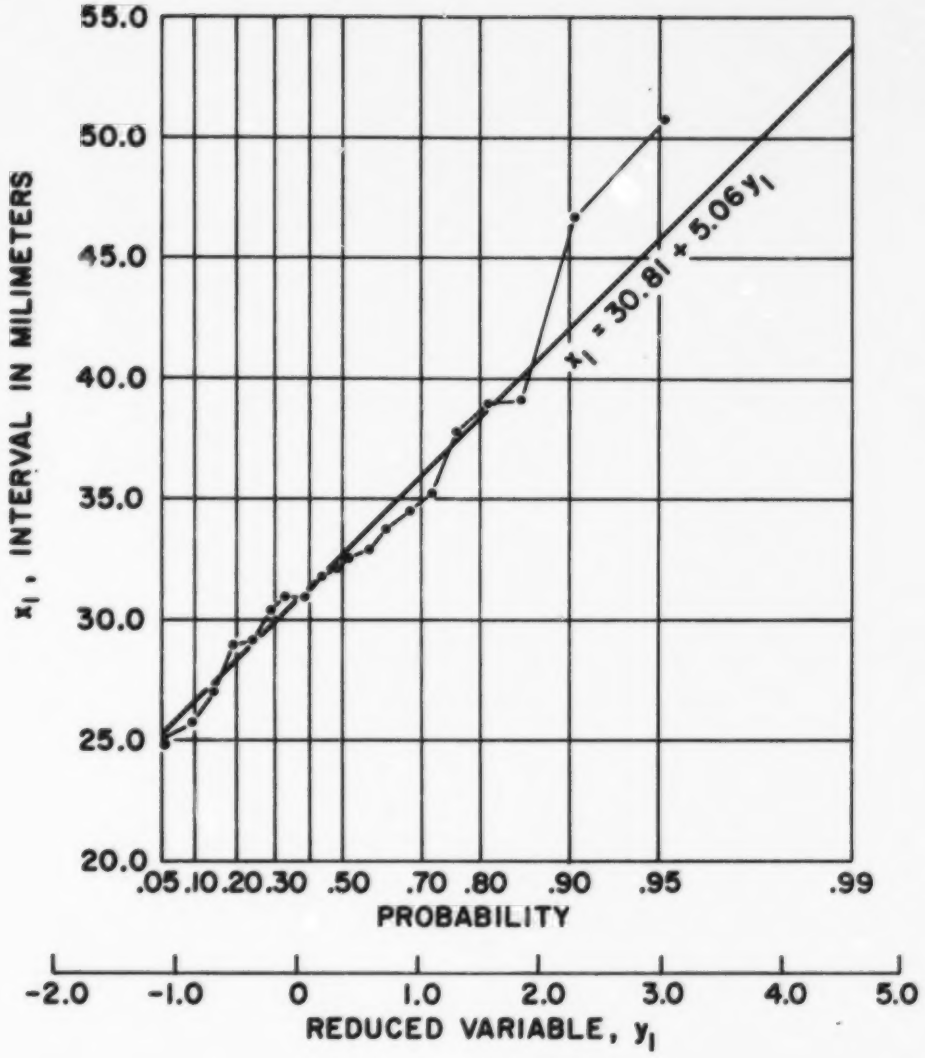

FIG. 1 SHOWING THE ANALYSIS OF THE FIRST EXTREME

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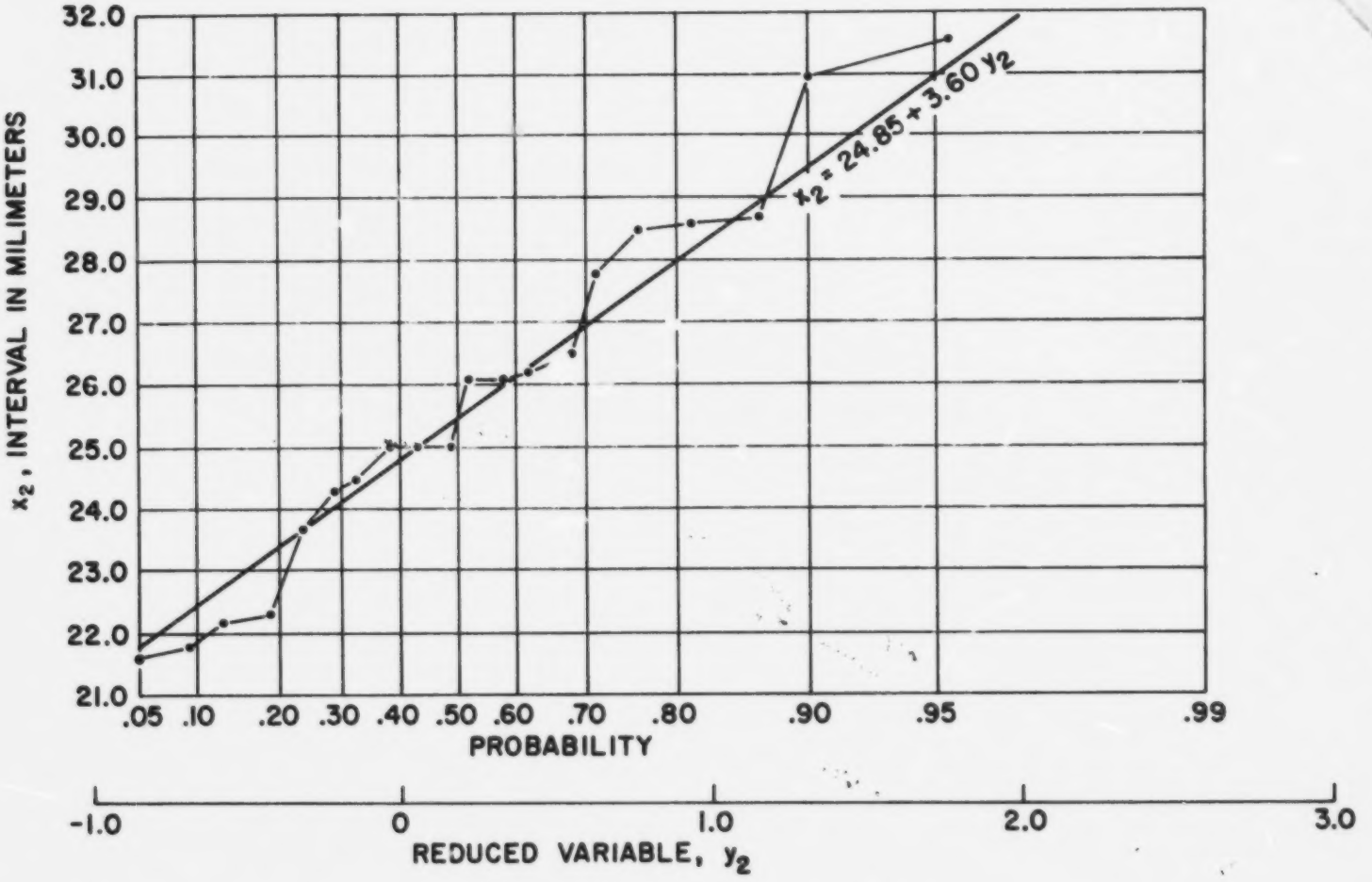

FIG. 2 SHOWING THE ANALYSIS OF THE SECOND EXTREME

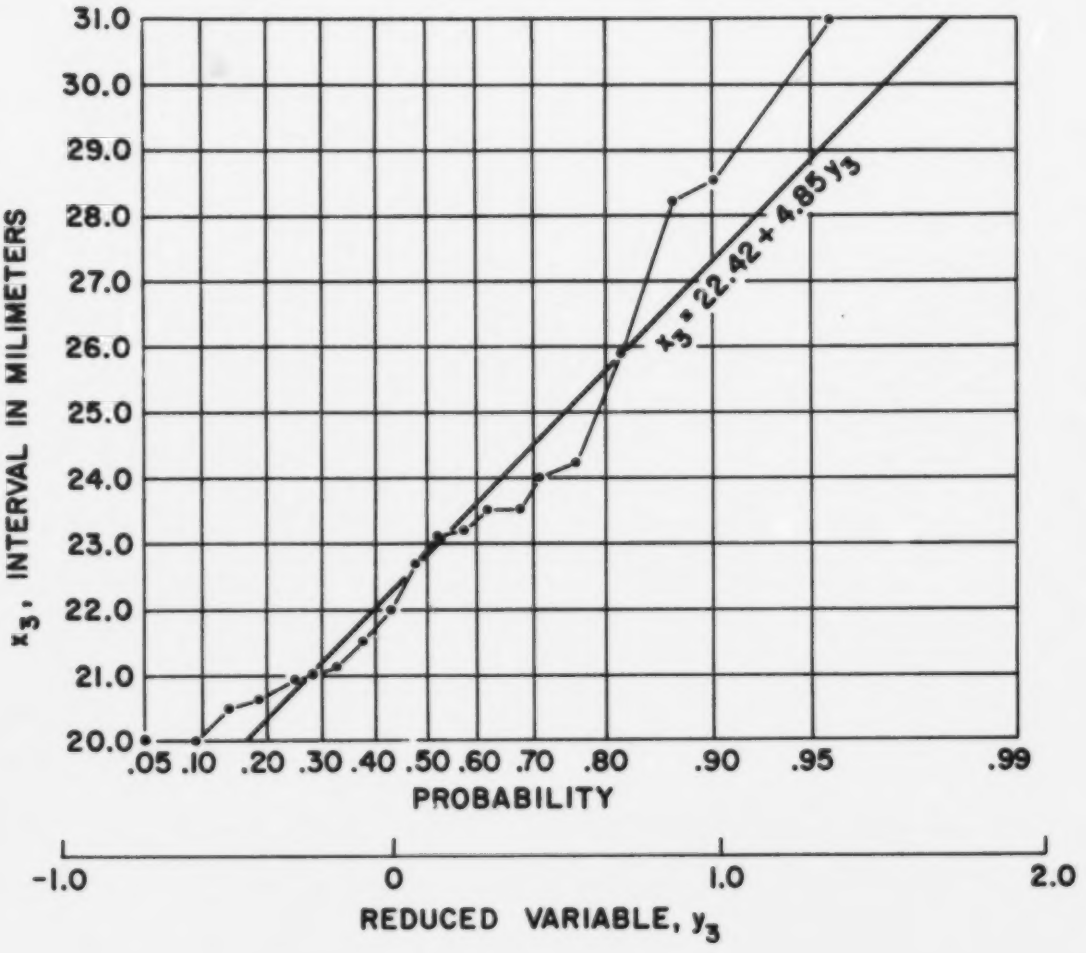

FIG. 3 SHOWING THE ANALYSIS OF THE THIRD EXTREME